

The minimum width condition for neutrino conversion in matter

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Abstract

We find that for small vacuum mixing angle θ and low energies ($s \ll M_Z^2$) the width of matter, $d_{1/2}$, needed to have conversion probability $P \geq 1/2$ should be larger than $d_{min} = \pi/(2\sqrt{2}G_F \tan 2\theta)$: $d_{1/2} \geq d_{min}$. Here G_F is the Fermi constant, s is the total energy squared in the center of mass and M_Z is the mass of the Z boson. The absolute minimum $d_{1/2} = d_{min}$ is realized for oscillations in a uniform medium with resonance density. For all the other density distributions (monotonically varying density, castle wall profile, etc.) the required width $d_{1/2}$ is larger than d_{min} . The width d_{min} depends on s , and for Z -resonance channels at $s \sim M_Z^2$ we get that $d_{min}(s)$ is 20 times smaller than the low energy value. We apply the minimum width condition, $d \geq d_{min}$, to high energy neutrinos in matter as well as in neutrino background. Using this condition, we conclude that the matter effect is negligible for neutrinos propagating in AGN and GRBs environments. Significant conversion can be expected for neutrinos crossing dark matter halos of clusters of galaxies and for neutrinos produced by cosmologically distant sources and propagating in the universe.

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1 Introduction

Since the paper by Wolfenstein [?], the neutrino transformations in matter became one of the most important phenomena in neutrino physics. Neutrinos propagating in matter undergo coherent forward scattering (refraction) described at low energies by the potential

$$V = \sqrt{2}G_F n , \quad (1)$$

where G_F is the Fermi constant, and n is a function of the density and chemical composition of the medium. For the case of $\nu_e - \nu_\mu$ and $\nu_e - \nu_\tau$ conversion in matter n coincides with the electron number density, n_e .

Refraction can lead to an enhancement of oscillations in media with constant density, and to resonant conversion in the varying density case (MSW effect) [?, ?]. For periodic, or quasi-periodic density profiles, various parametric effects can occur [?, ?, ?].

The MSW effect has been applied to solar neutrinos [?], and to neutrinos from supernovae [?]. Oscillations of neutrinos of various origins (solar, atmospheric, supernovae neutrinos, etc.) in the matter of the Earth have been extensively studied. Apart from resonance enhancement of oscillations, parametric effects are expected for neutrinos crossing both the mantle and the core of the Earth [?, ?, ?, ?]. The oscillations and conversion of active neutrinos into a sterile species can be important in the Early Universe [?]. Recently, matter effects on high energy neutrino fluxes from Active Galactic Nuclei (AGN) and Gamma Ray Bursters (GRBs) have been estimated [?]. Propagation of ultra-high energy neutrinos in halos of galaxies has been considered [?].

It is intuitively clear that to have a significant matter effect a sufficiently large amount of matter is needed. Let us define the width of the medium as the integrated density along the path travelled by the neutrino in the matter:

$$d = \int n_e(L) dL . \quad (2)$$

This quantity is frequently named “column density” in astrophysical context. We will show that there exists a minimum value d_{min} for the width below which it is not possible to have significant neutrino conversion. This lower bound is independent of the density profile and of the neutrino energy and mass. That allows us to make conclusions on the relevance of matter effect in various situations without knowledge of the density distribution.

The paper is organized as follows: in section 2 we derive the minimum width condition for the conversion in matter between two active neutrino flavours, and check it for different density profiles. In section 3 we discuss the generalizations of the condition to the active-sterile case and to conversion induced by flavour changing neutrino-matter interactions. We also study the matter effect in the small width limits. Section 4 presents a study of the minimum width condition for high energy neutrinos both in matter and in neutrino background. Section 5 is devoted to applications of our results to neutrino propagation in AGN and GRBs environments, in dark matter halos and in the Early Universe. Conclusions and discussion follow in section 6.

2 The minimum width condition

In this section we consider various mechanisms of matter enhancement of neutrino flavour conversion. For each of them we work out the minimum width of the medium needed to have significant conversion probability, showing that a lower bound for the width exists and is realized in the case of uniform medium with resonance density.

2.1 The absolute minimum width

Let us consider a system of two mixed flavour states¹, ν_e and ν_μ (ν_τ), characterized by vacuum mixing angle θ and mass squared difference Δm^2 . In a uniform medium the states oscillate and the transition probability, as a function of the distance L , is given by:

$$P_{\nu_e \rightarrow \nu_\mu}(L) = \sin^2 2\theta_m \sin^2 \left(\pi \frac{L}{l_m} \right), \quad (3)$$

where θ_m and l_m are the mixing angle and the oscillation length in the medium:

$$\begin{aligned} \sin 2\theta_m &= \frac{\sin 2\theta}{[(2EV/\Delta m^2 - \cos 2\theta)^2 + \sin^2 2\theta]^{1/2}} \\ l_m &= \frac{l}{[(2EV/\Delta m^2 - \cos 2\theta)^2 + \sin^2 2\theta]^{1/2}}. \end{aligned} \quad (4)$$

Here $l = 4\pi E/\Delta m^2$ is the vacuum oscillation length and E is the neutrino energy.

We assume that the vacuum mixing is small, so that vacuum oscillations effects are negligible ($P_{\nu_e \rightarrow \nu_\mu}^{vac} \ll 1$) and a strong transition in medium, i.e. $P_{\nu_e \rightarrow \nu_\mu} = O(1)$, is essentially due to matter effect. For definiteness, we choose the condition of significant conversion to be

$$P_{\nu_e \rightarrow \nu_\mu} \geq \frac{1}{2}. \quad (5)$$

Let us consider a uniform medium with resonance density [?]:

$$n_e^{res} = \frac{\Delta m^2}{2\sqrt{2}EG_F} \cos 2\theta. \quad (6)$$

In this case the oscillation amplitude is $\sin 2\theta_m = 1$, and the oscillation length equals

$$l^{res} = \frac{l}{\sin 2\theta} = \frac{4\pi E}{\Delta m^2 \sin 2\theta}. \quad (7)$$

According to eq. (3) the condition (5) starts to be satisfied for $L = l_m/4$, and the corresponding width is:

$$d_{min} = \frac{1}{4} n_e^{res} l^{res}. \quad (8)$$

¹The arguments remain the same for three neutrinos.

Inserting the expressions of n_e^{res} and l^{res} given in (6) and (7), we get:

$$d_{min} = \frac{\pi}{2\sqrt{2}G_F \tan 2\theta} = \frac{d_0}{\tan 2\theta} , \quad (9)$$

where

$$d_0 = \frac{\pi}{2\sqrt{2}G_F} \simeq \frac{1.11}{G_F} . \quad (10)$$

We will call d_0 the refraction width. Numerically,

$$d_0 = 2.45 \cdot 10^{32} \text{cm}^{-2} = 4.08 \cdot 10^8 \text{A cm}^{-2} , \quad (11)$$

where $A = 6 \cdot 10^{23}$ is the Avogadro number².

The widths d_{min} and d_0 have a simple physical interpretation. The refraction width d_0 is a universal quantity: it is determined only by the Fermi coupling constant, and does not depend on the neutrino parameters at all. Using the definition of refraction length

$$l_0 \equiv \frac{2\pi}{V} = \frac{2\pi}{\sqrt{2}n_e G_F} \quad (12)$$

we can write:

$$\frac{d_0}{n_e} = \frac{l_0}{4} . \quad (13)$$

It appears that d_0 corresponds to the distance at which the matter-induced phase difference between the flavour states equals $\pi/2$. This can be considered as the definition of refraction width, which by eq. (12) can be written in the general form:

$$d_0 \equiv \frac{\pi}{2} \frac{n_m}{V_m} , \quad (14)$$

where V_m is the neutrino-medium potential and n_m is the concentration of the relevant scatterers in the medium.

The minimum width, d_{min} , is inversely proportional to $\tan 2\theta$, which represents properties (the mixing) of the neutrino system itself. The smaller the mixing θ , the larger is the width d_{min} needed for strong transition.

²It can be checked that different choices of the condition (5) lead to analogous results. For instance, taking $P_{\nu_e \rightarrow \nu_\mu} \geq \frac{3}{4}$ we find

$$d_0^{3/4} = \frac{4}{3} d_0 = \frac{2\pi}{3\sqrt{2}G_F} = 5.41 \cdot 10^8 \text{A cm}^{-2} ,$$

and, for $P_{\nu_e \rightarrow \nu_\mu} \geq \frac{1}{4}$:

$$d_0^{1/4} = \frac{2}{3} d_0 = \frac{\pi}{3\sqrt{2}G_F} = 2.7 \cdot 10^8 \text{A cm}^{-2} .$$

The condition (5) can be generalized. It corresponds to the case of initial state coinciding with a pure flavour state. In general one can require that the change of the probability to detect a given flavour α is larger than $1/2$:

$$\Delta P \equiv P_f(\nu_\alpha) - P_i(\nu_\alpha) \geq \frac{1}{2} , \quad (15)$$

where P_i and P_f are the initial and final probabilities. The condition (5) corresponds to $P_i(\nu_\mu) = 0$, so that $P_f(\nu_\alpha) = P_{\nu_e \rightarrow \nu_\mu}$. Taking $P_i = 1/4$ and $P_f = 3/4$, we get in a similar way:

$$d_{1/2} = \frac{2}{3}d_{min} = \frac{\pi}{3\sqrt{2}G_F \tan 2\theta} . \quad (16)$$

This $d_{1/2}$ is the extreme value, however for most practical situations the condition (5) is more relevant, and from here on we will use the the width d_{min} determined in (9).

In what follows we will show that for all the other density profiles the width $d_{1/2}$ required by the condition (5) is larger than d_{min} .

2.2 Uniform medium with density out of resonance

For $n_e \neq n_e^{res}$ the inequality (5) can be satisfied only if $\sin^2 2\theta_m \geq \frac{1}{2}$, which means that the density is required to be in the resonance interval: $n_e^{res}(1 - \tan 2\theta) \leq n_e \leq n_e^{res}(1 + \tan 2\theta)$. At the edges of the interval we get the width

$$d_{1/2} = \frac{\pi}{2G_F} \left(\frac{1}{\tan 2\theta} \pm 1 \right) \simeq \sqrt{2}d_{min} , \quad (17)$$

which is larger than d_{min} . For other values of the density in the resonance interval we have $d_{min} < d_{1/2} < \sqrt{2}d_{min}$ ³.

2.3 Medium with varying density

In general the neutrino propagation has a character of interplay of resonance conversion and oscillations. Two conditions are needed for strong transition:

- 1) Resonance condition: the neutrinos should cross the layer with resonance density.
- 2) Adiabaticity condition: the density should vary slowly enough. This condition can be written in terms of the adiabaticity parameter γ at resonance:

$$\gamma \ll 1 \quad (18)$$

$$\gamma \equiv \frac{2E \cos 2\theta}{\Delta m^2 \sin^2 2\theta} \frac{1}{n_e} \left| \frac{dn_e}{dL} \right| . \quad (19)$$

³It can be checked that the width $d_{1/2}$ in eq. (17) is larger than d_{min} for small mixing: $\sin 2\theta \lesssim 0.3$. We will use this condition as criterion of smallness of the mixing.

Notice that both the conditions 1) and 2) are local, and can be fulfilled for arbitrarily small widths of the medium. Clearly, they are not sufficient to assure a significant conversion, and a third condition of large enough matter width is needed.

Let us consider a linear density profile with length $2L$ and average density equal to the resonance one, so that $n_{max} = n_e^{res} + \Delta n$ and $n_{min} = n_e^{res} - \Delta n$. Denoting θ_{1m} and θ_{2m} the mixings in the initial and final points, we find that in the first order of adiabatic perturbation theory the conversion probability is given by:

$$\begin{aligned} P_{\nu_e \rightarrow \nu_\mu}(L) &= \frac{1}{2} - \frac{1}{2} \cos 2\theta_{1m} \cos 2\theta_{2m} \\ &- \frac{1}{2} \sin 2\theta_{1m} \sin 2\theta_{2m} \cos \left[\frac{1}{\gamma} f(x) \right] \\ &- 2 \sin(2\theta_{1m} - 2\theta_{2m}) \alpha(x) \cos \left[\frac{1}{2\gamma} f(x) \right] , \end{aligned} \quad (20)$$

where

$$\begin{aligned} x &= 2\pi\gamma \frac{L}{l^{res}} , \\ f(x) &= \ln(x + \sqrt{1+x^2}) + x\sqrt{1+x^2} , \\ \alpha(x) &= \int_0^x \frac{dy}{1+y^2} \cos \left[\frac{1}{2\gamma} f(y) \right] . \end{aligned} \quad (21)$$

For $\gamma \ll 1$ we get from eqs. (20) and (21):

$$d_{1/2} = d_{min} \left[1 + \left(1 - \frac{\pi}{8} \right) \gamma^2 \right] . \quad (22)$$

This expression shows that for the adiabatic case $d_{1/2} \simeq d_{min}$ and for weak violation of adiabaticity the minimum width increases quadratically with γ . We remark that in this case the effect is dominated by oscillations with large (close to maximal) depth. The change of density gives only small corrections.

Let us consider now a situation in which the resonance adiabatic conversion is the main mechanism of flavour transition. A pure conversion effect is realized if the initial neutrino state that enters the medium coincides with one of the eigenstates of the Hamiltonian in matter, and the propagation in matter is adiabatic. In this case no phase effect, and therefore no oscillations occur. Let us denote n_i and n_f the initial and final densities of the medium, and suppose the initial state is $\nu_i = \nu_{2m} = \sin \theta_m \nu_e + \cos \theta_m \nu_\mu$. The probability to find a ν_μ in this state is $P_i(\nu_\mu) = \cos^2 \theta_m(n_i)$. The state evolves following the change of density, so that it remains an eigenstate of the Hamiltonian, and the probability to find ν_μ in the final state is $P_f(\nu_\mu) = \cos^2 \theta_m(n_f)$. Since the initial state ν_i does not coincide with a pure flavour state we will use the condition (15) as criterion of strong matter effect. Inserting P_i and P_f in (15), we get the condition for $d_{1/2}$:

$$\cos 2\theta_m(n_f) - \cos 2\theta_m(n_i) = 1 . \quad (23)$$

Taking the initial and final values of the density as $n_i = n_e^{res} + \Delta n$ and $n_f = n_e^{res} - \Delta n$ ($\Delta n \geq 0$), and using the definition (4) we find that the equality (23) leads to

$$\Delta n = n_e^{res} \frac{1}{\sqrt{3}} \tan 2\theta . \quad (24)$$

Clearly, for a given Δn the size of the layer, and therefore its width, depend on the gradient of the density which can be expressed in terms of the adiabaticity parameter γ , eq. (19). We get:

$$n_e(L)dL = \frac{2E \cos 2\theta}{\Delta m^2 \sin^2 2\theta} \frac{1}{\gamma} dn_e , \quad (25)$$

and then integrating this equation we obtain:

$$d = \frac{2E \cos 2\theta}{\Delta m^2 \sin^2 2\theta} \frac{1}{\gamma} \Delta n . \quad (26)$$

Finally, inserting the expressions (24) and (6) in eq. (26) we find:

$$d_{1/2} = \frac{4}{\pi \sqrt{3}} \frac{1}{\gamma} d_{min} . \quad (27)$$

Let us comment on this result. As far as the adiabaticity condition is satisfied, the change of probability does not depend on the density distribution; it is a function of the initial and final densities only. If Δn is fixed, the decrease of the width means the decrease of the length L of the layer, and therefore increase of the gradient of the density. This will lead eventually to violation of the adiabaticity condition. Thus, the minimal width corresponds to the maximal γ for which the adiabaticity is not broken substantially.

For strong adiabaticity violation an increase of $d_{1/2}$ is expected, due to the increase of the minimum Δn required by the condition (15), and therefore of the corresponding length. This can be seen if we consider the previous argument taking into account the effect of the adiabaticity breaking from the beginning. Using the Landau-Zener level crossing probability $P_{LZ} = \exp(-\pi/2\gamma)$, which describes the transition between two eigenstates, we get, instead of (23):

$$(1 - 2P_{LZ})(\cos 2\theta_m(n_f) - \cos 2\theta_m(n_i)) = 1 , \quad (28)$$

where we have averaged out the interference terms. Then instead of eq. (24) we get

$$\Delta n = n_e^{res} \frac{1}{\sqrt{16P_{LZ}^2 - 16P_{LZ} + 3}} \tan 2\theta . \quad (29)$$

Finally, the condition for $d_{1/2}$ can be written as:

$$d_{1/2} = \frac{4}{\pi \gamma \sqrt{16P_{LZ}^2 - 16P_{LZ} + 3}} d_{min} . \quad (30)$$

For $\gamma \rightarrow 0$ eq. (30) gives $d_{1/2} \rightarrow \infty$, according to the fact that the density changes very slowly and therefore the width needed to have significant conversion increases. With the increase of γ the width $d_{1/2}$ decreases and has a minimum at $\gamma \simeq 0.7$, for which we find $d_{1/2} \simeq 1.5 d_{min}$. With further increase of γ ($\gamma \gtrsim 0.7$) the width $d_{1/2}$ increases rapidly. According to (30) it diverges for $P \rightarrow 1/4$, when $\gamma \rightarrow \pi/(4 \ln 2) \simeq 1.13$. This value corresponds to the case in which the adiabaticity violation is so strong that even an infinite amount of matter is not enough to satisfy the condition (15). Thus, we have found that also in this case $d_{1/2} > d_{min}$.

2.4 Step-like profile

As an extreme case of strong adiabaticity violation, let us consider the profile consisting of two layers of matter, having densities $n_1 = n_e^{res} + \Delta n$ and $n_2 = n_e^{res} - \Delta n$ ($\Delta n \geq 0$), and equal lengths $L_1 = L_2 = L$. At the border between the layers the density has a jump of size $2\Delta n$. We fix $L = l^{res}/8$, so that $d = d_{min}$. The result for the conversion probability can be computed exactly:

$$P_{\nu_e \rightarrow \nu_\mu}^{step} = s^2 \sin^2 \left(\frac{\pi}{4s} \right) + s^2 c^2 \left[1 - \cos \left(\frac{\pi}{4s} \right) \right]^2, \quad (31)$$

where we denote the mixing parameters in the two layers as $\sin 2\theta_{2m} = \sin 2\theta_{1m} \equiv s$, $\cos 2\theta_{2m} = -\cos 2\theta_{1m} \equiv c$. In absence of the step ($\Delta n = 0$), $P_{\nu_e \rightarrow \nu_\mu}^{step}$ equals $1/2$, recovering the case $n_e = n_e^{res} = \text{const}$. The probability (31) decreases monotonically as Δn increases. Expanding in $\delta = (\Delta n / n_e^{res} \tan 2\theta)^2$ we get⁴:

$$P_{\nu_e \rightarrow \nu_\mu}^{step} \simeq \frac{1}{2} - (\sqrt{2} - 1 - \frac{\pi}{8})\delta \simeq \frac{1}{2} - 0.02 \delta. \quad (32)$$

According to (32), for $d = d_0$ we have $P_{\nu_e \rightarrow \nu_\mu}^{step} < 1/2$. This implies that, to have $P_{\nu_e \rightarrow \nu_\mu}^{step} = 1/2$ one needs $d_{1/2} > d_{min}$.

2.5 Castle-wall profile

The profile consists of a periodical sequence of alternate layers of matter, having two different densities n_1 and n_2 . We denote the corresponding mixing angles as θ_{1m} and θ_{2m} . In this case, a strong transformation requires certain conditions on the oscillation phases acquired by neutrinos in the layers [?]; therefore the transformation is a consequence of the specific density profile, rather than of an enhancement of the mixing. Suppose $n_1 \ll n_e^{res}$ and $n_2 = 0$, and take the width of each layer to be equal to half oscillation length, so that the oscillation phase acquired in each layer is π . It can be shown [?] that for small θ this is the condition under which the conversion probability increases most rapidly with the

⁴This approximation proves to be very good (relative error $\leq 0.5\%$) for $0 \leq \delta \leq 1$, i.e. for n_1 and n_2 in the resonance interval.

distance. As a function of the number N of periods (a period corresponds to two layers), the probability is given by [?, ?]:

$$P_{\nu_e \rightarrow \nu_\mu}(N) = \sin^2(2N\Delta\theta) , \quad (33)$$

where $\Delta\theta \equiv \theta_{1m} - \theta_{2m} = \theta_{1m} - \theta$. Using the approximation $2\theta_{1m} - 2\theta \simeq \sin 2\theta_{1m} - \sin 2\theta$, and expanding $\sin 2\theta_{1m}$ in n_1 , we get:

$$d_{1/2} = \frac{\pi^2}{2\sqrt{2}G_F} \frac{1}{\sin 2\theta} \simeq \pi d_{min} . \quad (34)$$

Again, we find that $d_{1/2} \geq d_{min}$.

Thus, for all the known mechanisms of matter enhancement of flavour transition (resonant oscillations, adiabatic conversion, parametric effects), we have found that the width $d_{1/2}$ is larger than d_{min} , which is realized for the case of uniform medium with resonance density. In fact the constant profile with resonance density could be expected from the beginning to represent an extreme case: this profile is singled out, since it is the simplest distribution with the density fixed at the unique value n_e^{res} .

It is worthwhile to introduce also the total nucleon width. Let us consider a medium made of electrons, protons and neutrons with number densities n_e , n_p and n_n . Defining the number of electrons per nucleon as $Y_e \equiv n_e/(n_n + n_p)$, we can write the total nucleon width that corresponds to d_0 as:

$$d_{0N} \equiv \frac{d_0}{Y_e} . \quad (35)$$

We can also introduce the total mass width d_ρ :

$$d_\rho \equiv m_N d_{0N} = \frac{m_N d_0}{Y_e} . \quad (36)$$

For electrically and isotopically neutral medium ($n_e = n_n = n_p$), eq. (35) gives:

$$d_{0N} = 2d_0 , \quad (37)$$

and numerically:

$$d_{0N} = 4.9 \cdot 10^{32} \text{ cm}^{-2} , \quad d_\rho = 8.16 \cdot 10^8 \text{ g} \cdot \text{cm}^{-2} . \quad (38)$$

3 Generalizations

In this section we generalize the previous results to active-sterile transition and to the case of flavour-changing induced conversion. We also discuss the small width limits.

3.1 Active-sterile conversion

In this case the scattering both on electrons and on nucleons contributes to the conversion, and the effective potential for an electron neutrino in an electrically neutral medium equals

$$V = \sqrt{2}G_F n_e \left(1 - \frac{n_n}{2n_p}\right). \quad (39)$$

Thus, the results for $\nu_e - \nu_s$ transition can be obtained from those for $\nu_e - \nu_\mu$ by the replacement $n_e \rightarrow n_e(1 - n_n/2n_p)$. For the refraction width we get immediately:

$$d_0(\nu_e \rightarrow \nu_s) = d_0 \left| \frac{2n_p}{2n_p - n_n} \right|. \quad (40)$$

In particular, for an isotopically neutral medium eq. (40) gives

$$d_0(\nu_e \rightarrow \nu_s) = 2d_0. \quad (41)$$

Notice that, for highly neutronized media ($n_n \gg n_p$) the width $d_0(\nu_e \rightarrow \nu_s)$ gets significantly smaller than d_0 . In this case, however, the physical situation is more properly described by the total nucleon width d_{0N} defined in eq. (35), since the effect is mainly due to the scattering on neutrons. We find:

$$d_{0N}(\nu_e \rightarrow \nu_s) = 2d_0 \left| \frac{n_p + n_n}{2n_p - n_n} \right|, \quad (42)$$

which gives in the limit $n_n \rightarrow \infty$:

$$d_{0N}(\nu_e \rightarrow \nu_s) = 2d_0, \quad (43)$$

similarly to eq. (41).

For the $\nu_\mu - \nu_s$ case, the potential, and consequently the width, can be obtained by replacing $n_e \rightarrow n_e(-n_n/2n_p)$, which gives

$$d_0(\nu_\mu \rightarrow \nu_s) = d_0 \frac{2n_p}{n_n}. \quad (44)$$

For isotopically neutral medium, eq. (44) reduces to eq. (41). For highly neutronized media, the argument is analogous to the one for the $\nu_e - \nu_s$ case, and we get the same result as in eq. (43).

3.2 Oscillations induced by flavour changing (FC) neutrino-matter interactions

In this case the neutrino masses can be zero, or negligible, and the flavour transition is a pure matter effect. The Hamiltonian of the system has the following form [?]:

$$H = \sqrt{2}G_F \begin{pmatrix} 0 & \epsilon n_f \\ \epsilon n_f & \epsilon' n_f \end{pmatrix}, \quad (45)$$

where n_f is the effective number density of the scatterers, and ϵ and ϵ' are parameters of the interaction.

As follows from eq. (45), in a uniform medium the neutrinos oscillate with transition probability:

$$P_{\nu_e \rightarrow \nu_\mu}(L) = \frac{4\epsilon^2}{4\epsilon^2 + \epsilon'^2} \sin^2 \left(\pi \frac{L}{l} \right), \quad (46)$$

$$l = \frac{\pi\sqrt{2}}{\sqrt{4\epsilon^2 + \epsilon'^2}} \frac{1}{G_F n_f}. \quad (47)$$

We assume $\epsilon' < \epsilon$, which is needed to have a significant oscillation amplitude. Using eq. (8), we get:

$$\begin{aligned} d_{1/2}^{FC} &= \frac{\pi}{2\sqrt{2}G_F} \frac{1}{\sqrt{4\epsilon^2 + \epsilon'^2}} \frac{n_e}{n_f} \\ &= \frac{d_0}{\sqrt{4\epsilon^2 + \epsilon'^2}} \frac{n_e}{n_f}. \end{aligned} \quad (48)$$

Notice that the factor $(n_e/n_f) \tan 2\theta / \sqrt{4\epsilon^2 + \epsilon'^2}$ implies that $d_{1/2}^{FC}$ can be significantly smaller than d_{min} , and the oscillation effect can be observed in media of smaller width. For a FC neutrino interaction with up (or down) quarks and isotopically neutral medium we have $n_e/n_f = 1/3$, and therefore:

$$d_{1/2}^{FC} \simeq \frac{1}{6\epsilon} d_0. \quad (49)$$

Notice that there are two sources of decrease of the width: the factor 2 is given by the presence of two off diagonal terms in the Hamiltonian (45) and the factor 3 is due to the larger number of scatterers. Taking $\epsilon = 1$, we find the value:

$$d_{1/2}^{FC} \simeq 1.36 \cdot 10^8 \text{ g} \cdot \text{cm}^{-2}. \quad (50)$$

For a density $n = 4 \text{ g} \cdot \text{cm}^{-3}$ (Earth's crust), this corresponds to the distance $L = 337 \text{ Km}$, which is comparable to the length of the present long base-line neutrino experiments: K2K (base-line 250 Km) and ICANOE (740 Km).

3.3 The small width limits

In a number of situations (see section 5) the width of the medium is smaller, or much smaller, than d_{min} . We consider, then, the matter effect on oscillations in the limit $d/d_{min} \ll 1$. Introducing the two variables:

$$\lambda \equiv \frac{L}{l^{res}/4} \quad \rho \equiv \frac{n_e}{n_e^{res}}, \quad (51)$$

we can write the small width condition as:

$$d/d_{min} = \lambda\rho \ll 1. \quad (52)$$

We focus on two important realizations of this inequality:

1) Small size of the layer and density close to resonance. As we have shown in section 2, a strong transition requires $n_e \simeq n_e^{res}$. In case of small width this implies small size of the layer. Therefore we have $\lambda \ll 1$ and $\rho \sim 1$. In this case $d/d_{min} \simeq \lambda$. We expand the oscillation probability (3) in λ at the lowest (nonzero) order, and find that the matter effect vanishes quadratically with d/d_{min} :

$$P(\lambda, \rho) \simeq \left(\frac{\pi}{4}\right)^2 \lambda^2 \sim \left(\frac{\pi}{4}\right)^2 \left(\frac{d}{d_{min}}\right)^2. \quad (53)$$

2) Small density of the medium and length close to the minimum value $l^{res}/4$. Another condition of strong conversion is to have the size of the layer of the order of the oscillation length. According to eq. (52), this means that the density is small. Thus, we have $\rho \ll 1$ and $\lambda \sim 1$, and therefore $d/d_{min} \simeq \rho$. In order to give a phase-independent description of the matter effect, we perform an expansion in ρ of the oscillation amplitude:

$$\sin^2 2\theta_m - \sin^2 2\theta = 2\rho \sin^2 2\theta \cos 2\theta \sim 2\frac{d}{d_{min}} \sin^2 2\theta \cos 2\theta. \quad (54)$$

Unlike the previous case, the relative matter effect is linear in d/d_{min} .

4 Refraction of high energy neutrinos

In this section we examine the refraction of high energy neutrinos ($s \gtrsim M_Z^2$), both in matter and in neutrino background.

4.1 High energy neutrinos in matter

Let us consider the propagation of high energy neutrinos in medium composed of protons, neutrons and electrons. The expressions (1) and (10) refer to the low energy range, $s \ll M_W^2$, where s is the center of mass energy squared of the incoming neutrino and the target electron, and M_W is the mass of the W boson. The general formulas, valid for high energies too, can be obtained by restoring the effect of the complete propagator of the W boson in the expression of the potential (an analogous argument holds for the Z boson).

Let us consider $\nu_e - \nu_\mu$ conversion. Since the refraction effects are determined by the real part of the propagator, the potential (1) is generalized as:

$$V = \sqrt{2}G_F n_e f(q_W^2) \quad (55)$$

$$f(q_W^2) \equiv \frac{1 - q_W^2}{(1 - q_W^2)^2 + \gamma_W^2}, \quad (56)$$

where $q_W^2 \equiv q^2/M_W^2$ and $\gamma_W \equiv \Gamma_W/M_W$; q and Γ_W are the four momentum and the width of the W boson.

The only contribution to the potential (55) is given by the forward charged current scattering on electrons (u -channel exchange of W), for which $q^2 \simeq -s$. Therefore, introducing $s_W \equiv s/M_W^2$, we have $q_W^2 \simeq -s_W$. From the potential (55) we can find the refraction width d_0 using the definition (14). For the nucleon refraction width (35) we find:

$$d_{0N}(s_W) = \frac{1}{Y_e} \frac{d_0}{f(-s_W)} = \frac{d_0}{Y_e} (1 + s_W) , \quad (57)$$

where we have neglected the width γ_W . Eq. (57) shows that $d_{0N}(s_W)$, and therefore $d_{min}(s_W)$, increase linearly with s_W above the threshold of the W boson production.

For the active-sterile conversion, one has to take into account also the neutral current interaction channel (t -channel exchange of Z), for which $q^2 = 0$, so that the low energy formulas (1-10) are still valid. For $\nu_\mu - \nu_s$ only neutral current processes are involved, thus the low energy result, eq. (44), holds at high energies too. In contrast, for the $\nu_e - \nu_s$ case both charged and neutral current interactions contribute, and for an electrically neutral medium the high energy potential can be written as:

$$V = \sqrt{2} G_F n_e \left(\frac{1}{1 + s_W} - \frac{n_n}{2n_p} \right) . \quad (58)$$

The second term in eq. (58) does not depend on s_W , thus coinciding with the corresponding term in the low energy expression (39). The potential (58) gives the nucleon refraction width:

$$d_{0N}(s_W) = 2d_0 \left| \frac{1 + s_W}{(3Y_e - 1) - s_W(1 - Y_e)} \right| . \quad (59)$$

For isotopically neutral medium ($Y_e = 1/2$) we get:

$$d_{0N}(s_W) = 4d_0 \left| \frac{1 + s_W}{1 - s_W} \right| . \quad (60)$$

The width $d_{0N}(s_W)$ diverges for $s_W \rightarrow 1$ (see fig. 1).

At high energies inelastic interactions and absorption become important: at $s_W \sim 1$ the imaginary part of the interaction amplitude is comparable with the real part. In fig. 1 we show the refraction width $d_\rho = m_N d_{0N}$ for $\nu_e - \nu_\mu$ and $\nu_e - \nu_s$ conversion and the absorption width d_{abs} [?, ?] as functions of the neutrino energy E in the rest frame of the matter. We have considered isotopically neutral medium, $Y_e = 1/2$. The absorption width d_{abs} is dominated by the contribution of neutrino-nucleon scattering; it decreases monotonically with the energy E . In contrast, d_ρ starts to increase at $s_W \sim 1$, which corresponds to $E = 10^6 \div 10^7$ GeV, according to eq. (57). For $E \simeq 10^6$ GeV absorption and refraction become comparable; at higher energies, the former effect dominates: $d_{abs} \lesssim d_0$. This means that for a ν_e with energy $E > 10^6$ GeV the conversion in matter is damped by inelastic interactions and absorption [?, ?, ?], therefore a strong conversion effect can not be expected.

Notice that for small mixing angle θ the minimum width d_{min} is significantly larger than the refraction width d_0 , therefore the absorption starts to be important at lower energies. Taking, for instance, $\sin 2\theta = 0.3$ we have $d_{min} \simeq d_0/\sin 2\theta \simeq 3.3d_0$, and find that $d_{abs} \lesssim d_{min}$ already for $E \gtrsim 5 \cdot 10^5$ GeV.

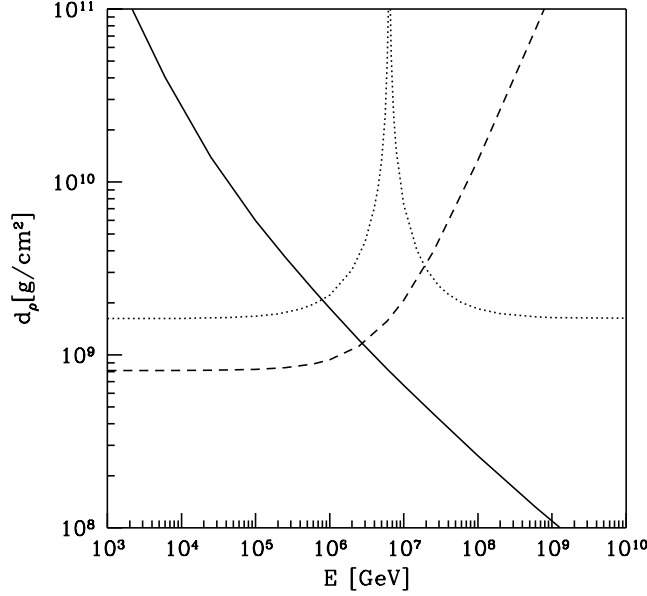


Figure 1: The width $d_\rho = m_N d_{0N}$ for $\nu_e - \nu_\mu$ (dashed line) and for $\nu_e - \nu_s$ (dotted line) channels, and the absorption width, d_{abs} , for the electron neutrino (solid line), as functions of the neutrino energy. We have considered isotopically neutral medium, $Y_e = 0.5$. The data for d_{abs} are taken from ref. [?].

Let us consider now the matter effect for conversion of antineutrinos. For $\bar{\nu}_e - \bar{\nu}_\mu$ channel the only contributing interaction is the $\bar{\nu}_e - e$ scattering with W exchanged in the s -channel. In this case $q^2 = s$, and using eq. (56) we get:

$$d_{0N}(s_W) = \frac{1}{Y_e} \frac{d_0}{|f(s_W)|} . \quad (61)$$

This function has a pole at $s_W = 1$, i.e., in the W -boson resonance. The pole appears because the amplitude becomes purely imaginary in the resonance, so that the potential is zero. The width $d_{0N}(s_W)$ diverges for $s_W \rightarrow \infty$, due to the $1/s_W$ decrease of the amplitude. The function (61) has two minima:

$$d_{0N}(s_W^{min}) = 2\gamma_W \frac{d_0}{Y_e} = 2\gamma_W d_{0N}(s_W = 0) \quad \text{at} \quad s_W^{min} = 1 \pm \gamma_W. \quad (62)$$

Numerically $d_{0N}(s_W^{min}) = 0.05 d_{0N}(s_W = 0)$, which shows that refraction effects are enhanced close to the W resonance. However, in this region inelastic interactions become already important.

For $\bar{\nu}_e - \bar{\nu}_s$ channel the contribution of neutrino-nucleon scattering should be included, and for electrically neutral medium we find:

$$d_{0N}(s_W) = 2d_0 \left| \frac{(1 - s_W)^2 + \gamma_W^2}{(3Y_e - 1) + 2s_W(1 - 2Y_e) - s_W^2(1 - Y_e) - \gamma_W^2(1 - Y_e)} \right|. \quad (63)$$

In the case of isotopically neutral matter eq. (63) gives:

$$d_{0N}(s_W) = 4d_0 \left| \frac{(1 - s_W)^2 + \gamma_W^2}{1 - s_W^2 - \gamma_W^2} \right|, \quad (64)$$

which has the value $4d_0$ in the limits $s_W \ll 1$ and $s_W \gg 1$, and a pole at $s_W \simeq 1$. Similarly to eq. (62) we find the minima:

$$d_{0N}(s_W^{min}) = 4\gamma_W d_0 = \gamma_W d_{0N}(s_W = 0) \quad \text{at} \quad s_W^{min} = 1 \pm \gamma_W. \quad (65)$$

In fig. 2 we show the refraction width $d_\rho = m_N d_{0N}$ for $\bar{\nu}_e - \bar{\nu}_\mu$ and $\bar{\nu}_e - \bar{\nu}_s$ channels and the absorption width for the electron antineutrino, d_{abs} [?], as functions of the neutrino energy. We have considered isotopically neutral medium. For energies outside the W boson resonance interval the main contribution to d_{abs} is given by the neutrino-nucleon scattering; at $s_W \simeq 1$ the effect of the resonant $\bar{\nu}_e - e$ scattering dominates, providing the narrow peak. It appears that absorption prevails over refraction ($d_{abs} < d_0$) for $E \gtrsim 6 \cdot 10^6$ GeV, corresponding to $d_\rho \simeq 6 \cdot 10^7 \text{g} \cdot \text{cm}^{-2}$, for both $\bar{\nu}_e - \bar{\nu}_\mu$ and $\bar{\nu}_e - \bar{\nu}_s$ cases.

The effect of absorption on neutrino conversion starts to be important at lower energies: for $\sin 2\theta = 0.3$ we find that $d_{abs} \lesssim d_{min}$ at $E \gtrsim 6 \cdot 10^5$ GeV.

4.2 High energy neutrinos in neutrino environment

Let us consider a beam of neutrinos which propagates in a background made of neutrinos of very low energies⁵. This could be the case of beams of low energy neutrinos from supernovae, or high energy neutrinos from AGN and GRBs, or neutrinos produced by the annihilation of superheavy relics, etc.. We assume that the background consists of neutrinos and antineutrinos of various flavours, with number densities n_i ($i = \nu_e, \bar{\nu}_e, \nu_\mu$, etc.). In the case of relativistic neutrino background we assume its isotropy.

The potential for a neutrino ν_α ($\alpha = e, \mu, \tau$) due to neutrino-neutrino interaction can be written as:

$$V_{\nu_\alpha}(s_Z) = \sqrt{2}G_F \left[n_{\nu_\alpha} f(-s_Z) - n_{\bar{\nu}_\alpha} f(s_Z) + \sum_{i=\nu_e, \nu_\mu, \nu_\tau} (n_i - n_{\bar{i}}) \right], \quad (66)$$

⁵We will not consider the conversion of neutrinos in the background itself, which can significantly affect the flavour content of the background.

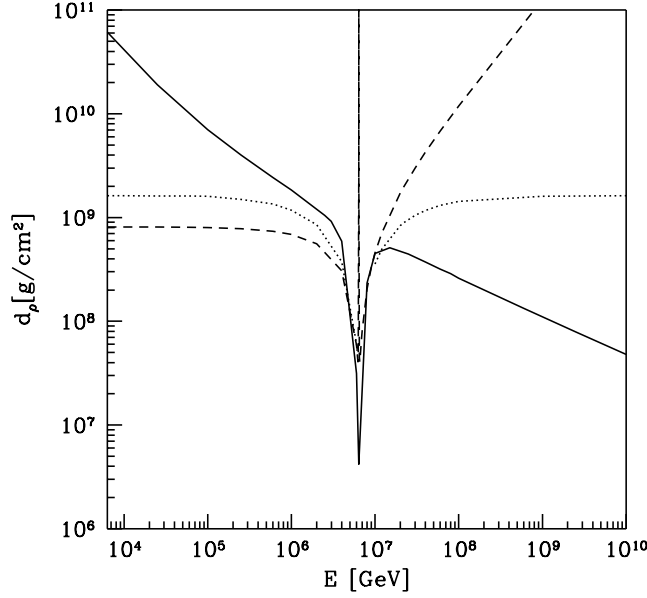


Figure 2: The width $d_\rho = m_N d_{0N}$ for $\bar{\nu}_e - \bar{\nu}_\mu$ (dashed line) and for $\bar{\nu}_e - \bar{\nu}_s$ (dotted line) channels, and the absorption width, d_{abs} , for the electron antineutrino (solid line), as functions of the neutrino energy. We have considered isotopically neutral medium, $Y_e = 0.5$. The data for d_{abs} are taken from ref. [?].

where the propagator function $f(s_Z)$ has been defined in eq. (56). Here $s_Z \equiv s/M_Z^2$ and $\gamma_Z \equiv \Gamma_Z/M_Z$; M_Z and Γ_Z are the mass and width of the Z -boson. The first term in eq. (66) is due to $\nu_\alpha - \nu_\alpha$ scattering with Z -boson exchange in u -channel, and the second term is the contribution from $\nu_\alpha - \bar{\nu}_\alpha$ annihilation.

4.3 Neutrino conversion in CP-asymmetric neutrino background

As a first case we consider a strongly CP-asymmetric neutrino background, and suppose $n_i \gg n_{\bar{i}}$, so that we can neglect the contributions of antineutrinos in (66). For simplicity, we assume equal concentrations for the neutrino species: $n_{\nu_e} = n_{\nu_\mu} = n_{\nu_\tau}$. In terms of the total number density of neutrinos, $n_\nu \equiv n_{\nu_e} + n_{\nu_\mu} + n_{\nu_\tau}$, the potential (66) reduces to:

$$V_{\nu_\alpha}(s_Z) = \sqrt{2}G_F n_\nu \left[1 + \frac{1}{3}f(-s_Z) \right]. \quad (67)$$

The potential for the antineutrino is given by $V_{\bar{\nu}_\alpha}(s_Z) = -V_{\nu_\alpha}(-s_Z)$.

Let us now find the refraction width d_0 and d_{min} for various channels.

1). For the active-sterile conversion, $\nu_\alpha - \nu_s$, the potential (67) coincides with the difference of the potentials for the two species, and therefore, by eq. (14), it gives immediately the refraction width of neutrinos:

$$d_0(s_Z) = d_0 \left| 1 + \frac{1}{3} f(-s_Z) \right|^{-1}. \quad (68)$$

The width $d_0(s_Z)$ is constant for $s_Z \ll 1$ and $s_Z \gg 1$: $d_0(s_Z \ll 1) \simeq 3d_0/4 = 1.84 \cdot 10^{32} \text{cm}^{-2}$, and $d_0(s_Z \gg 1) \simeq d_0$ (see fig.3).

Let us now compare the refraction and absorption effects. The main contribution to the absorption width d_{abs} [?] is given by the $\nu_\alpha - \nu_\alpha$ and $\nu_\alpha - \nu_\beta$ ($\beta \neq \alpha$) scatterings. The width d_{abs} decreases monotonically with s_Z and at $s_Z \gg 1$ it takes the value $d_{abs}(s_Z \gg 1) \simeq \pi/(2G_F^2 M_Z^2) \simeq 3.6 \cdot 10^{33} \text{cm}^{-2}$. Due to its non-resonant behaviour, d_{abs} is larger than d_0 for any energy of the neutrinos: at $s_Z \gg 1$ we find that $d_0/d_{abs} \simeq G_F M_Z^2/\sqrt{2} = \pi\alpha_W/(2\cos^2\theta_W) \simeq 0.1$, where θ_W and $\alpha_W = g^2/4\pi$ are the weak mixing angle and coupling constant. Therefore, d_{abs} is also larger than the minimum width, d_{min} , for $\sin 2\theta \gtrsim d_0/d_{abs} \simeq 0.1$.

2). For the conversion of an active antineutrino into a sterile species, $\bar{\nu}_\alpha - \bar{\nu}_s$, we get the width:

$$d_0(s_Z) = d_0 \left| 1 + \frac{1}{3} f(s_Z) \right|^{-1}. \quad (69)$$

This function (see fig.3) has a resonant behaviour with minima at $s_Z \simeq 1 \pm \gamma_Z$: $d_0(1-\gamma_Z) \simeq (1/6\gamma_Z + 1)^{-1}d_0 \simeq d_0/7$ and $d_0(1+\gamma_Z) \simeq (1/6\gamma_Z - 1)^{-1}d_0 \simeq d_0/5$. Outside the Z -boson resonance $d_0(s_Z)$ is constant: $d_0(s_Z \ll 1) \simeq 3d_0/4$ and $d_0(s_Z \gg 1) \simeq d_0$. In the range $s_Z \sim 1$ inelastic scattering and absorption become important. We evaluate the absorption width d_{abs} for antineutrino in neutrino background using the plots in ref. [?]. For $s_Z < 1$, the width d_{abs} decreases with the increasing s_Z ; at $s_Z \simeq 1$ it shows the characteristic peak due to the resonant $\nu_\alpha - \bar{\nu}_\alpha$ scattering. For $s_Z \gtrsim 1$ the absorption width increases with s_Z up to the limit $d_{abs}(s_Z \gg 1) \sim 3 \cdot 10^{33} \text{cm}^{-2}$, due to the contributions of $\nu_\beta - \bar{\nu}_\alpha$ scatterings ($\beta \neq \alpha$) and $\nu_\alpha - \bar{\nu}_\alpha$ interaction in the t -channel. We find that $d_{abs} \gtrsim d_0(s_Z)$ for $s_Z \lesssim 0.8$ ($s \lesssim 7 \cdot 10^3 \text{GeV}^2$) and for $s_Z \gtrsim 2$ ($s \gtrsim 1.7 \cdot 10^4 \text{GeV}^2$). Furthermore, $d_0(s_Z = 0.8) \simeq 10^{32} \text{cm}^{-2}$ and $d_0(s_Z = 2) \simeq 4 \cdot 10^{32} \text{cm}^{-2}$. Taking $\sin 2\theta = 0.3$ we get that $d_{abs} \gtrsim d_{min}$ for $s_Z \lesssim 0.7$ and $s_Z \gtrsim 2.4$; $d_{min}(s_Z = 0.7) \simeq 3 \cdot 10^{32} \text{cm}^{-2}$ and $d_{min}(s_Z = 2.4) \simeq 10^{33} \text{cm}^{-2}$.

Notice that, in contrast with the conversion in matter (see figs.1 and 2), for neutrinos and antineutrinos in neutrino environment we can have $d_{abs} \gtrsim d_{min}(s_Z)$ even in the high energy range, $s_Z \gtrsim 1$: in particular, we find that $d_{abs}(s_Z \gg 1) \gtrsim d_{min}(s_Z \gg 1)$ for $\sin 2\theta \gtrsim 0.1$.

3). Let us now consider the active-active conversion, $\nu_\alpha - \nu_\beta$. Assuming equal concentrations of neutrinos of different flavours, $n_{\nu_e} = n_{\nu_\mu} = n_{\nu_\tau}$, we find from eq. (67) that the

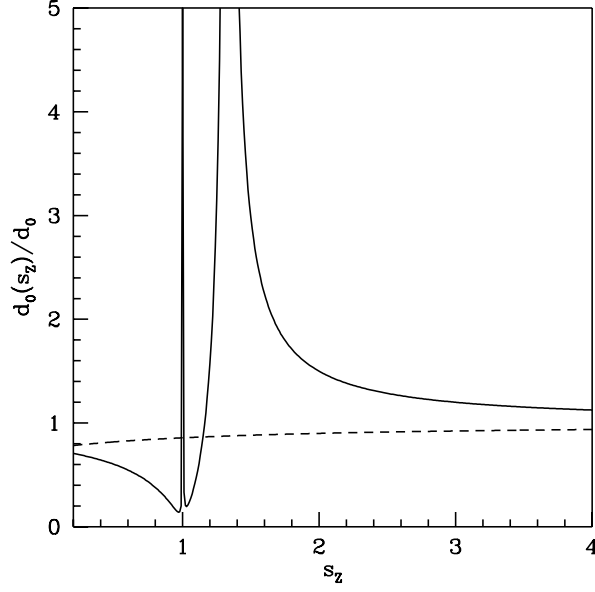


Figure 3: The dependence of the refraction width d_0 for $\nu_\alpha - \nu_s$ (dashed line) and $\bar{\nu}_\alpha - \bar{\nu}_s$ (solid line) channels on s_Z in strongly CP-asymmetric background, $n_i \gg n_{\bar{i}}$. Equal concentrations are assumed for the various flavours.

difference of the potentials of the two species equals:

$$\Delta V_{\alpha,\beta} = V_{\nu_\alpha} - V_{\nu_\beta} = \frac{1}{3}\sqrt{2}G_F n_\nu \left[f(-s_Z^\alpha) - f(-s_Z^\beta) \right] , \quad (70)$$

where $s_Z^i \equiv s^i/M_Z^2$ ($i = \alpha, \beta$), and s^i is the center of mass energy squared of the incoming and the background neutrino of the same type i . If the species ν_α and ν_β in the background have different energies we get that $s_Z^\alpha \neq s_Z^\beta$, and therefore $\Delta V_{\alpha,\beta} \neq 0$, leading to matter induced neutrino conversion even if ν_α and ν_β have equal concentrations. This situation is realized if the background neutrinos ν_α and ν_β have different masses, e.g. $m_{\nu_\alpha} > m_{\nu_\beta}$, and are non-relativistic. Denoting by E the energy of the neutrino beam, in the rest frame of the background we have $s^i = 2m_i E$, and thus $s_Z^\alpha/s_Z^\beta = m_{\nu_\alpha}/m_{\nu_\beta} > 1$. The condition $s_Z^\alpha \neq s_Z^\beta$ is achieved also if one of the neutrino species is relativistic and the other is not: $m_{\nu_\alpha} \gg E_\beta \gg m_{\nu_\beta}$, where E_β is the energy of ν_β in the background. Assuming the isotropy of the neutrino gas, we have that $s^\beta \simeq 2E_\beta E$.

Using (70) and (14), we get the refraction width:

$$d_0(s_Z^i) = 3d_0 \left| f(-s_Z^\alpha) - f(-s_Z^\beta) \right|^{-1} \simeq 3d_0 \left| \frac{(1 + s_Z^\alpha)(1 + s_Z^\beta)}{(s_Z^\alpha - s_Z^\beta)} \right| . \quad (71)$$

The function (71) diverges for $s_Z^i \rightarrow \infty$ and $s_Z^i \rightarrow 0$. In particular, in the low energy limit, $s_Z \ll 1$, it reduces to $d_0(s_Z^i \ll 1) \simeq 3d_0/(2E\Delta m)$, where $\Delta m \equiv m_{\nu_\alpha} - m_{\nu_\beta}$. For the realistic case $s_Z^\alpha \gtrsim 1$ and $s_Z^\beta \ll 1$, eq. (71) can be written as:

$$d_0(s_Z^\alpha) \simeq 3d_0 \left| \frac{1 + s_Z^\alpha}{s_Z^\alpha} \right|, \quad (72)$$

which approaches the minimum value $3d_0$ when $s_Z^\alpha \gg 1$ (see fig.4). Taking the maximal realistic values for the mass and energy of the neutrino, $m_{\nu_\alpha} = 5$ eV and $E = 10^{22}$ eV we get $s_Z^\alpha \simeq 12$ at most, so that $d_0(s_Z^i) \simeq 3.5d_0$.

4). For the $\bar{\nu}_\alpha - \bar{\nu}_\beta$ channel the effective potential equals:

$$\Delta V_{\alpha,\beta} = \frac{1}{3}\sqrt{2}G_F n_\nu \left[f(s_Z^\alpha) - f(s_Z^\beta) \right], \quad (73)$$

and therefore we get the width:

$$d_0(s_Z^i) = 3d_0 \left| f(s_Z^\alpha) - f(s_Z^\beta) \right|^{-1}. \quad (74)$$

Due to the resonant character of the function $f(s_Z)$, the width $d_0(s_Z^i)$ has the following features (see fig.4):

- (i) It reaches the local minimum $d_0(s_Z^i) \simeq 6\gamma_Z d_0 \sim d_0/6$ when one of the s_Z^i 's is at resonance and the other is outside the resonance: $s_Z^\alpha \simeq 1$ and $s_Z^\beta \neq 1$ (or vice versa).
- (ii) The absolute minimum $d_0(s_Z^i) \simeq 3\gamma_Z d_0$ is achieved when $s_Z^\alpha \simeq 1 + \gamma_Z$ and $s_Z^\beta \simeq 1 - \gamma_Z$ (or vice versa). These conditions can be satisfied for certain relations between the masses of the background neutrinos. For non-relativistic background: $m_{\nu_\alpha}/m_{\nu_\beta} = (1 + \gamma_Z)/(1 - \gamma_Z)$.

Notice that for s_Z^i discussed in (i) and (ii) the effects of inelastic scattering and absorption can be important.

- (iii) If one of the s_Z^i 's is far below the resonance and the other is far above (e.g. $s_Z^\alpha \gg 1$ and $s_Z^\beta \ll 1$) then $d_0(s_Z^i) \sim 3d_0$.

- (iv) $d_0(s_Z^i) \gg d_0$ if both the s_Z^i 's are far below or far above the resonance.

Obviously, for strong CP-asymmetric background with $n_{\bar{i}} \gg n_i$ the results for ν and $\bar{\nu}$ channels should be interchanged.

4.4 Neutrino conversion in CP-symmetric neutrino background

Let us now consider the neutrino conversion in a CP-symmetric neutrino background, $n_i = n_{\bar{i}}$, with $n_{\nu_e} = n_{\nu_\mu} = n_{\nu_\tau}$. In this case the potential (66) can be written as:

$$V_{\nu_\alpha} = \sqrt{2}G_F n_{\nu_\alpha} [f(-s_Z) - f(s_Z)]. \quad (75)$$

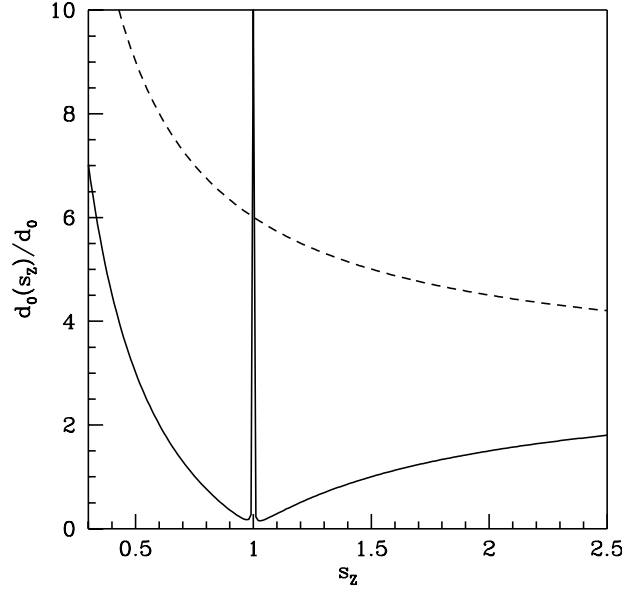


Figure 4: The dependence of the refraction width d_0 for $\nu_\alpha - \nu_\beta$ (dashed line) and $\bar{\nu}_\alpha - \bar{\nu}_\beta$ (solid line) channels on s_Z in strongly CP-asymmetric background, $n_i \gg n_{\bar{i}}$. Equal concentrations are assumed for the various flavours. We have considered $s_Z^\alpha \gg s_Z^\beta \sim 0$.

It vanishes in the low energy limit $s_Z \rightarrow 0$, but it is unsuppressed at high energies, $s_Z \gtrsim 1$, leading to significant matter effect. We will consider the conversion of neutrinos; due to the CP-symmetry antineutrinos undergo analogous effects.

1). For $\nu_\alpha - \nu_s$ conversion from the potential (75) we find the refraction width for the neutrinos of flavour α :

$$\begin{aligned} d_0(s_Z) &= d_0 |f(-s_Z) - f(s_Z)|^{-1} \\ &= d_0 \left| \frac{(1 + s_Z^2 + \gamma_Z^2)^2 - 4s_Z^2}{2s_Z[(1 - s_Z^2) + \gamma_Z^2]} \right|. \end{aligned} \quad (76)$$

For $s_Z \ll 1$ it behaves as $d_0(s_Z) \simeq d_0/(2s_Z)$ and for $s_Z \gg 1$ we have $d_0(s_Z) \simeq d_0 s_Z/2$ (see fig.5). The width (76) has two minima:

$$d_0(s_Z^{min}) \simeq 2\gamma_Z d_0 \quad \text{at} \quad s_Z^{min} = 1 \pm \gamma_Z. \quad (77)$$

Numerically, $d_0(s_Z^{min}) = 0.055 d_0 \simeq 1.35 \cdot 10^{31} \text{cm}^{-2}$.

The absorption width d_{abs} is dominated by $\nu_\alpha - \bar{\nu}_\alpha$ annihilation, with a resonance peak at $s_Z \sim 1$. Using the results of ref. [?] we find that d_{abs} is larger than $d_0(s_Z)$ outside the Z -boson resonance, and the two quantities are comparable at $s_Z \sim 1$ or at $s_Z \gg 1$.

For $\sin 2\theta = 0.3$ we find that the minimum width d_{min} is larger than d_{abs} in the range $0.7 \lesssim s_Z \lesssim 1.6$, corresponding to $6 \cdot 10^3 \text{GeV}^2 \lesssim s \lesssim 1.3 \cdot 10^4 \text{GeV}^2$. At the edges of this interval the width d_{min} takes the value $d_{min}(s_Z = 0.7) \simeq d_{min}(s_Z = 1.6) \simeq 3 \cdot 10^{32} \text{cm}^{-2}$.

2). For the $\nu_\alpha - \nu_\beta$ channel, eq. (75) gives the difference of potentials:

$$\begin{aligned} \Delta V_{\alpha,\beta} &= V_{\nu_\alpha} - V_{\nu_\beta} \\ &= \sqrt{2} G_F n_{\nu_\alpha} \{ [f(-s_Z^\alpha) - f(s_Z^\alpha)] - [f(-s_Z^\beta) - f(s_Z^\beta)] \} . \end{aligned} \quad (78)$$

The corresponding refraction width equals:

$$d_0(s_Z^i) = d_0 \left| [f(-s_Z^\alpha) - f(s_Z^\alpha)] - [f(-s_Z^\beta) - f(s_Z^\beta)] \right|^{-1} . \quad (79)$$

We find that $d_0(s_Z^i) \gtrsim d_0$ when both s_Z^α and s_Z^β are outside the Z -boson resonance and $d_0(s_Z^i)$ takes its minimum values when either s_Z^α or s_Z^β is close to the Z -resonance:

If $s_Z^\alpha \sim 1 + \gamma_Z$ and $s_Z^\beta \sim 1 - \gamma_Z$ (or vice versa) $d_0(s_Z^i)$ has the absolute minimum $d_0(s_Z^i) \simeq \gamma_Z d_0$. For $s_Z^\alpha \sim 1$ and $s_Z^\beta \neq 1$ (or vice versa) we have the local minimum $d_0(s_Z^i) \simeq 2\gamma_Z d_0$. Notice that in the realistic case $s_Z^\alpha \gg s_Z^\beta \sim 0$ the width (79) reduces essentially to the one in eq. (76). At resonance, where $d_0(s_Z^i)$ has minima, the effects of inelastic collisions and absorption are important.

5 Applications

The results derived in the previous sections are now applied to some physical situations of interest. After a brief discussion of well known cases, like the Earth, the Sun and supernovae, we present results for neutrinos in some new astrophysical environments. We find that significant matter induced conversion can be expected for neutrinos crossing the dark matter halos of clusters of galaxies and for neutrinos from cosmologically distant sources.

5.1 Minimum width condition and bounds on the mixing

As follows from the analysis in section 2, a significant neutrino conversion in matter requires the fulfilment of the minimum width condition⁶:

$$d \geq d_{min} = \frac{d_0}{\tan 2\theta} . \quad (80)$$

This condition is independent of the density distribution, and therefore of the specific matter effect involved. Thus the knowledge of the width d allows one to conclude about the significance of the matter effect even if the density profile is unknown. This is the

⁶This condition refers to the requirement of conversion probability larger than 1/2, eq. (5). In some circumstances, however, even a small effect, with conversion probability $P \ll 1/2$ can be important.

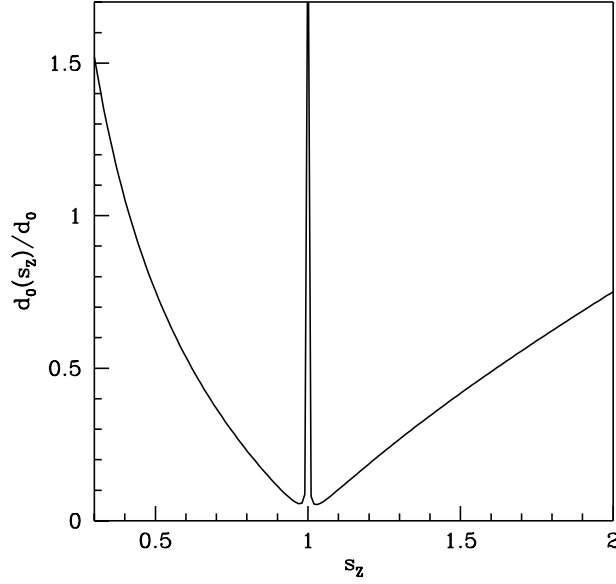


Figure 5: The dependence of the refraction width d_0 for $\nu_\alpha - \nu_s$ and $\bar{\nu}_\alpha - \bar{\nu}_s$ channels on s_z in CP-symmetric neutrino background.

case of some astrophysical objects for which estimates or bounds on d can be obtained directly by observational data with no assumption on their internal structure.

In the Table 1 we show the parameters of interest of some objects, together with the values of the ratio

$$r \equiv \frac{d}{d_0} . \quad (81)$$

For $r < 1$, and small mixing angle, the condition (80) can not be satisfied, thus no significant neutrino conversion is expected. Conversely, for $r \gtrsim 1$, (80) can be fulfilled and gives the bound on the mixing:

$$\sin 2\theta \gtrsim \frac{1}{r} = \frac{d_0}{d} . \quad (82)$$

Notice that our analysis holds for small mixings: $\sin 2\theta \ll 1$. For applications we assume $\sin 2\theta \lesssim 0.3$, for which we find from eq. (82) that the minimum width condition is satisfied for $r \gtrsim 3$.

The inequality (82) can be considered as the sensitivity limit for the mixing angle that can be achieved by studies of neutrino conversion in a layer of given width d . The real sensitivity can be however much lower than the absolute limit given by the condition (82). This is related to the fact that in the case of varying density only part of the total amount

of matter effectively contributes to the conversion. Introducing the corresponding width d^{conv} we have the condition:

$$\sin 2\theta \gtrsim \frac{d_0}{d^{conv}} , \quad (83)$$

instead of the (82).

Let us find the expression of d^{conv} for a medium with monotonically varying density. As discussed in section 2.3, the transition occurs mainly in the resonance layer. Using the result (24) we get:

$$d^{conv} = n_e^{res} \frac{dL}{dn} 2\Delta n = \frac{2}{\sqrt{3}} n_e^{res} l_n \tan 2\theta , \quad (84)$$

where $l_n \equiv |(\frac{dn}{dL})^{-1}|_{res} n_e^{res}$. Inserting the expression (84) in the condition (83), we find:

$$\sin^2 2\theta \gtrsim \frac{\sqrt{3}d_0}{2n_e^{res} l_n} . \quad (85)$$

Clearly, d^{conv} could be much smaller than the total width d of the object, so that the condition (85) on the mixing could be much stronger than (82). Notice that the bound (85) is quadratic in $\sin 2\theta$. Using the definition (19) of the adiabaticity parameter, γ , the condition (85) can be written as $\gamma \leq 4/(\pi\sqrt{3})$, which corresponds to the adiabaticity condition close to its limit of validity.

Another important issue is that the maximal sensitivity for the mixing θ can be achieved for particular values of $\Delta m^2/E$, which depend on the specific density profile. As follows from (85), for constant (or slowly varying with the distance) l_n the smallest $\sin^2 2\theta$ corresponds to the largest n_e^{res} , and therefore to the largest values of $\Delta m^2/E$. This is the case of exponential density profile. For power-law profile, $n_e \sim L^{-k}$, we get $|l_n| = L/k$, so that $\sin^2 2\theta \sim L^{k-1}$. Taking $k > 1$, fulfilled by practically all the realistic profiles, we find that the smallest θ is achieved for the smallest L , and consequently the highest values of n_e and $\Delta m^2/E$.

Notice that d^{conv} is a local property which depends on the derivative in l_n . Of course, the description given by d^{conv} is not correct when the density profile is close to the constant one, so that $l_n \rightarrow \infty$. In this case d^{conv} can be even larger than the total width d . Thus, the correct condition on the mixing can be written as:

$$\sin 2\theta \gtrsim \frac{d_0}{\min[d, d^{conv}]} . \quad (86)$$

5.2 The Sun, the Earth, the Moon and supernovae

For neutrinos crossing the Earth we consider two types of trajectories, corresponding to different values of the zenith angle θ_z . For $\cos \theta_z = 1$ neutrinos travel along the diameter of the Earth, crossing the core and the two layers of the mantle. We get $r = 13.6$, and therefore according to (82) we could expect significant matter conversion for $\sin^2 2\theta \gtrsim 5 \cdot 10^{-3}$. However this maximal sensitivity, which would be achieved for uniform density

object	density (cm ⁻³)	size (cm)	$r = d/d_0$
Earth:			
$\cos \theta_z = 1$	$2.6 \cdot 10^{24}$	$1.26 \cdot 10^9$	13.6
$\cos \theta_z = 0.81$	$1.5 \cdot 10^{24}$	10^9	6.4
Sun	$\sim 7 \cdot 10^{24}$	$6.96 \cdot 10^{10}$	2600
Moon	$\sim 10^{24}$	$3.48 \cdot 10^8$	1.4
Supernova	$3 \cdot 10^{33}$	10^7	10^9
Universe ($n_\nu = n_{\bar{\nu}}$)	$1.5 \cdot 10^4$	10^{27}	$3 \cdot 10^{-2}$
Universe ($n_\nu \gg n_{\bar{\nu}}$)	$\sim 10^5$	10^{27}	0.3
Galactic halo	$\lesssim 2 \cdot 10^6$	$3 \cdot 10^{23}$	$5 \cdot 10^{-2}$
Cluster halo	$\lesssim 5 \cdot 10^7$	$3 \cdot 10^{24}$	10
AGN		$d \simeq 10^{22} \div 10^{23} \text{cm}^{-2}$	$10^{-10} \div 10^{-9}$
GRB	$10^{10} \div 10^{12}$	$< 5 \cdot 10^{15}$	$< 10^{-5}$

Table 1: The density, the size and the matter width in units of refraction width, $r = d/d_0$, for various physical objects. The values given for the densities are averaged along the trajectories of the neutrinos. We quote the number density of electrons for objects made of usual matter, and the concentration of the neutrino background for the halos and the universe. For the Earth the results are given for two trajectories with different zenith angle θ_z . The results for the universe correspond to redshift $z = 5$ for the cases of $\nu_\alpha - \nu_s$ and $\bar{\nu}_\alpha - \bar{\nu}_s$ in CP-symmetric and strongly CP-asymmetric neutrino background with $\eta_\nu \simeq 1$.

distribution, is not realized for the Earth profile. For small mixing, the difference between the densities in the core and in the mantle is larger than the resonance interval. As a result, the oscillations are resonantly enhanced either in the mantle or in the core, and only one of the two parts effectively contributes to the effect. At the same time, for certain ranges of $\Delta m^2/E$, different from both the resonance values in the core and in the mantle, parametric enhancement of oscillations occurs. Numerical calculations [?] give $\sin^2 2\theta \gtrsim 2 \cdot 10^{-2}$ as best sensitivity.

For $\cos \theta_z = 0.81$ the trajectory is tangential to the core, and therefore it represents the path of maximal length in the mantle. In this case we find $r \simeq 6.4$ and the sensitivity limit $\sin^2 2\theta \gtrsim 2.5 \cdot 10^{-2}$. Since this case realizes approximatively the optimal condition of uniform medium, we have good agreement with the results of exact calculations.

In the case of the Moon, $r = 1.4$, and therefore a large mixing is required: $\sin^2 2\theta \gtrsim 0.5$.

A numerical integration of the density profile of the Sun [?] gives $d \simeq 1.5 \cdot 10^{12} \text{g} \cdot \text{cm}^{-2}$. Dividing this result by $d_\rho = m_N d_{0N}$, with $Y_e = 0.7$, we find $r \simeq 2600$. From the condition (82) we get then $\sin^2 2\theta \gtrsim 1.5 \cdot 10^{-7}$. This bound is remarkably weaker than the one obtained from the condition (85): taking $n_e^{res} \simeq 50 A \text{cm}^{-3}$ and $l_n \simeq 0.3 R_\odot$, we get $\sin^2 2\theta \simeq 2.4 \cdot 10^{-4}$, in good agreement with the results of exact computations [?, ?].

For supernovae the total width of the matter above the neutrinosphere gives $r \simeq 10^9$,

for which the condition (82) would lead to $\sin^2 2\theta \gtrsim 10^{-18}$. Using the density profile $n_e = n_e^0 (R_0/R)^3$ [?], with $R_0 = 10^7$ cm and $n_e^0 \simeq 10^{34} \text{cm}^{-3}$, from (85) we find $\sin^2 \theta \gtrsim 10^{-8}$, which agrees well with the results of numerical calculations [?].

As shown in the previous examples, the maximal sensitivity for $\sin^2 2\theta$, given by the total width d , can be achieved in the case of uniform medium at $\Delta m^2/E$ corresponding to the resonance density. Such a situation is realized for neutrinos crossing the mantle of the Earth. In the case of substantial deviations from the constant density, like in the Sun or in supernovae, the sensitivity is much lower. The stronger the deviation from constant density, the smaller d^{conv} , and therefore the lower is the sensitivity.

5.3 AGN and GRBs

Let us now turn to high energy neutrinos from Active Galactic Nuclei (AGN) and Gamma Ray Bursters (GRBs) [?, ?, ?, ?].

In AGN, neutrinos are considered to be produced by the interaction of accelerated protons with a photon or proton background [?, ?, ?]. There is a hope that neutrinos from AGN with energies $\gtrsim 10^6$ GeV could be detected by large scale underwater (ice) and EAS detectors [?].

The width of matter crossed by neutrinos in an AGN can be estimated on the basis of the existing data on the X-ray emission of these objects. The variability of the spectra suggests that the X-radiation is emitted very close to the AGN core [?]. The proton acceleration and therefore the neutrino production are supposed to happen in the same region. For this reason the width of matter crossed by neutrinos equals approximatively the one crossed by the X-radiation. For the later the experimental data [?, ?, ?] give the value $d_{AGN} \simeq (10^{-2} \div 10^{-1}) A \text{ cm}^{-2}$, therefore significant neutrino conversion in AGN is excluded⁷ (for a short discussion, see also ref. [?]).

A rather successful description of the origin of GRB is provided by the fireball model [?], in which neutrino production is predicted to happen in an analogous way as in AGN [?, ?]. A fireball can emit protons, detected as high-energy cosmic rays on Earth, accompanied by a flux of neutrinos. The requirement that the fireball should be transparent to protons gives an estimate of the width of the object: $d_{GRB} \leq d_{abs}$, where $d_{abs} = 10 \div 100 A \text{ cm}^{-2}$ is the total absorption width for the protons. It is possible to evaluate the width in a different way. An estimate of the electron number density in the fireball is given in ref. [?]: $n_{GRB} \simeq (10^{10} \div 10^{12}) \text{ cm}^{-3}$. Using this value, and taking the fireball mass in the range of star-like objects, $M = (1 \div 10) M_\odot$, we can get the radius of the object, $R_{GRB} = 5 \cdot (10^{14} \div 10^{15}) \text{ cm}$, and then the width: $d_{GRB} = 10 \div 10^4 A \text{ cm}^{-2}$. In agreement with the first argument, we see, then, that also in GRBs the matter effect on neutrino conversion is negligible.

⁷In the present discussion we have considered radial propagation of neutrinos from the inner to the external regions of the object. We have not considered neutrinos travelling through the core of the AGN. In this case a significant matter-induced conversion could occur, however neutrinos crossing the core are supposed to be a small fraction of the total neutrino flux produced.

5.4 Dark matter halos

According to models [?], part of the dark matter in halos of galaxies and clusters of galaxies should consist of neutrinos⁸. Therefore neutrinos of extragalactic origin crossing the halo on the way to the Earth undergo refraction on the neutrino background. It was suggested in ref. [?] that, due to non uniform neutrino density distribution in the galactic halo, ultrahigh energy neutrinos can be resonantly converted into active and sterile species.

Following ref. [?] we consider a galactic halo composed of non-relativistic neutrinos and antineutrinos of the two species ν_μ and ν_τ . The electron neutrino is assumed to be lighter, and therefore less clustered, than ν_μ and ν_τ : $n_{\nu_e} \ll n_i$, $i = \nu_\mu, \nu_\tau$. We assume CP-symmetry of the background: $n_i = n_{\bar{i}}$. We take the density profile [?]:

$$n_\nu(r) = n_\nu^0 \frac{1}{1 + (r/a)^2} , \quad (87)$$

where $n_\nu \equiv n_{\nu_\mu} + n_{\nu_\tau}$ and a is the core radius of the halo. For galactic halos this radius is estimated to be $a \simeq 10 \div 100$ kpc. An upper bound for n_ν^0 is given by the Tremaine-Gunn condition [?]: for identical fermions of mass m , the maximum number density n^{max} allowed by the Pauli principle equals

$$n^{max} = \frac{1}{6\pi^2} (mv^{esc})^3 , \quad (88)$$

where v^{esc} is the escape velocity of the particle:

$$v^{esc} = \left(\frac{2GM}{R} \right)^{\frac{1}{2}} \sim 540 \left(\frac{M}{10^{12} M_\odot} \right)^{\frac{1}{2}} \left(\frac{R}{30 \text{ Kpc}} \right)^{-\frac{1}{2}} \text{ Km/s.} \quad (89)$$

Here M and R are the total mass and radius of the galaxy. In what follows we will take $R \simeq 3a$, corresponding to the radius at which the density reduces at one tenth of its core value, $n_\nu(3a) = n_\nu^0/10$. From (88) and (89) we get:

$$n^{max} = 1.7 \cdot 10^6 \left(\frac{m_\nu}{5 \text{ eV}} \right)^3 \left(\frac{M}{10^{12} M_\odot} \right)^{\frac{3}{2}} \left(\frac{a}{10 \text{ Kpc}} \right)^{-\frac{3}{2}} \text{ cm}^{-3}. \quad (90)$$

The integration of the profile (87) gives the matter width:

$$d = \frac{\pi}{2} n_\nu^0 a . \quad (91)$$

Inserting the expression (90) for n^{max} in (91), we find the upper bound for the width:

$$d \lesssim 8 \cdot 10^{28} \left(\frac{m_\nu}{5 \text{ eV}} \right)^3 \left(\frac{M}{10^{12} M_\odot} \right)^{\frac{3}{2}} \left(\frac{a}{10 \text{ Kpc}} \right)^{-\frac{1}{2}} \text{ cm}^{-2} . \quad (92)$$

⁸In what follows, we will not consider the heavy particles present in the halos, because their number density is much smaller than the one of neutrinos, although they provide the largest part of the mass of the halo. Furthermore, the amplitude of the forward scattering of neutrinos on neutrinos and on the heavy particles of dark matter are comparable, or the former is even larger.

According to (92) the largest values of d are achieved for objects with big mass M and small radius a ; so that compact halos represent the most favourable case.

Let us now check the minimum width condition for $\nu_\mu - \nu_s$ and $\nu_\mu - \nu_e$ conversion in galactic halos. We use the refraction width $d_0(s_Z^{min}) = 0.055d_0 \simeq 1.35 \cdot 10^{31} \text{cm}^{-2}$ given in eq. (77). This value was obtained for $\nu_\mu - \nu_s$ conversion in CP-symmetric neutrino background, and is the absolute minimum of $d_0(s_Z)$ (eq. (76)), realized at the Z -boson resonance, $s_Z \sim 1$. Notice that, under the assumption $n_{\nu_e} \ll n_i$, the result $d_0(s_Z^{min})$ holds also for $\nu_\mu - \nu_e$ conversion: due to the absence of electron neutrinos in the background, $n_{\nu_e} \simeq 0$, the potential (75) for ν_e is negligible, and therefore the electron neutrino behaves as a sterile species.

From eq. (92) we see that, for typical values of M and a of a galaxy, like for instance the Milky Way ($M \simeq 10^{12} M_\odot$ and $a \simeq 10 \text{kpc}$), the minimum width condition is not satisfied: $d/d_0(s_Z^{min}) \lesssim 5 \cdot 10^{-3}$. For the galaxy M87 ($M \simeq 10^{13} M_\odot$ and $a \simeq 100 \text{kpc}$) we find $d/d_0(s_Z^{min}) \lesssim 5 \cdot 10^{-2}$. Taking an hypothetical very massive and compact object, with $M \simeq 10^{13} M_\odot$ and $a \simeq 10 \text{kpc}$ we get $d/d_0(s_Z^{min}) \lesssim 0.2$. Thus, a significant neutrino conversion effect in the galactic halo is excluded, in contrast with the result in ref. [?].

Conversely, significant matter-induced conversion can be realized in halos surrounding a cluster of galaxies. Taking the mass of a cluster as $M = (10^{13} \div 10^{15}) M_\odot$ and the size $a \simeq 1 \text{Mpc}$, we obtain from eq. (92)

$$d \lesssim 10^{29} \div 3 \cdot 10^{32} \text{cm}^{-2} \simeq (10^{-2} \div 20) d_0(s_Z^{min}) . \quad (93)$$

For the maximal value, $d/d_0(s_Z^{min}) \simeq 20$, from the condition (82) we get the sensitivity to the mixing: $\sin^2 2\theta \gtrsim 3 \cdot 10^{-3}$. With this value we find that the adiabaticity condition (18) is fulfilled for $\Delta m^2 \gtrsim 4 \cdot 10^{-6} \text{eV}^2$.

The maximal sensitivity is achieved in the energy range of the Z -resonance, where also inelastic scattering and absorption are important. Indeed, in section 4.4, taking $\sin 2\theta = 0.3$, we have found that $d_{abs} \sim d_{min}$ at $s \simeq 6 \cdot 10^3 \text{GeV}^2$, where $d_{min} \simeq 3 \cdot 10^{32} \text{cm}^{-2}$. This value coincides with the upper edge of the interval (93). For smaller $\sin 2\theta$ d_{min} is larger, and the absorption effect on oscillations becomes even more important. Notice that $d_0(s_Z)$ takes its minimum value at the Z -resonance: for neutrino energies outside the resonance $d_0(s_Z)$ is larger, so that $d/d_0(s_Z) < 1$ and the minimum width condition is not satisfied.

Thus, we have found that in the halos of clusters of galaxies a significant matter-induced conversion can be achieved in the narrow interval of energies of the Z -boson resonance, where, however, the absorption and the effects of inelastic interactions are important.

5.5 Early Universe

In this section we consider neutrinos produced by cosmologically distant sources and propagating in the universe. The refraction occurs due to the interaction of the neutrinos with the particle background of the universe made of neutrinos electrons and nucleons.

The number densities of baryons, n_b , and electrons, n_e , are given by $n_b = n_e = \eta_b n_\gamma$, where $\eta_b = 10^{-10} \div 10^{-9}$ is the baryon asymmetry of the universe and n_γ is the concentration of photons. At present time $n_\gamma = n_\gamma^0 \simeq 400 \text{ cm}^{-3}$. We will describe the neutrino background by the total number density $n_\nu + n_{\bar{\nu}}$ and the CP-asymmetry $\eta_\nu \equiv (n_\nu - n_{\bar{\nu}})/n_\gamma$. The value of η_ν is unknown. A natural assumption would be $\eta_\nu \simeq \eta_b$: in this –almost CP-symmetric– case the total concentration of neutrinos equals $n_\nu + n_{\bar{\nu}} = 4n_\gamma/11$ [?]. However strong asymmetry, $\eta_\nu \sim 1$, is not excluded. The upper bound $\eta_\nu \sim 10$ for muon and tau neutrinos follows from the Big Bang Nucleosynthesis [?, ?].

For $\eta_\nu \sim 1$ the contribution of the neutrino background to refraction dominates and the interaction of neutrinos with electrons and nucleons can be neglected. It can be checked that the contributions of neutrino-electron and neutrino-nucleon scattering to the refraction width are smaller than d_0 at any time after the neutrino decoupling epoch, $t_{dec} \simeq 1 \text{ s}$.

In the framework of the standard Big-Bang cosmology, the number density of neutrinos in the universe decreases with the increasing time as [?]:

$$n(t) = \begin{cases} n_0 \left(\frac{t_0}{t} \right)^2 & t \geq t_{eq} \\ n_{eq} \left(\frac{t_{eq}}{t} \right)^{\frac{3}{2}} & t < t_{eq} \end{cases} \quad (94)$$

Here $t_0 \simeq 10^{18} \text{ s}$ is the age of the universe, and $t_{eq} \simeq 10^{12} \text{ s}$ is the time at which the energy densities of radiation and of matter in the universe were approximatively equal. We denote by n_0 and n_{eq} the neutrino concentrations at $t = t_0$ and $t = t_{eq}$ respectively.

The matter width $d(t)$ crossed by the neutrinos from the time t of their production to the present one is given by the integration of the concentration (94):

$$d(t) = \int_t^{t_0} n(\tau) d\tau = \begin{cases} d_U \left[\frac{t_0}{t} - 1 \right] & t \geq t_{eq} \\ d_{eq} \left(\frac{t_{eq}}{t} \right)^{\frac{1}{2}} & t < t_{eq} \end{cases} \quad (95)$$

where $d_U \equiv t_0 n_0$ is the present width of the universe and $d_{eq} = d_U [t_0/t_{eq} - 1]$ is the width at $t = t_{eq}$.

In what follows we will focus on the case of matter domination epoch, $t \geq t_{eq}$, for which the width (95) can be expressed in terms of the redshift, $z \equiv (t_0/t)^{2/3} - 1$, as:

$$d(z) = d_U \left[(z+1)^{\frac{3}{2}} - 1 \right] = d_i \left[1 - (z+1)^{-\frac{3}{2}} \right] \quad (96)$$

where $d_i = t n = d_U (z+1)^{\frac{3}{2}}$ is the width of the universe at the time t of production of the neutrino beam; n is the concentration of the neutrino background at t . According to eq. (96), for large enough z the width at the production time, d_i , gives the dominant contribution.

Another important feature of the propagation of neutrinos from cosmological sources is the redshift of energy. The refraction width d_0 depends on the center of mass energy squared s of the incoming and the background neutrinos. As a consequence of the redshift, $s_Z = s_Z(z)$, the width d_0 changes with time (with z) during the neutrino propagation: $d_0 = d_0(s_Z(z))$. Thus, the width of matter d should be compared with some effective (properly averaged) refraction width d_0 which in fact depends on the channel of transition. For non-relativistic background neutrinos of mass m_ν we have that s increases with z as: $s \simeq 2m_\nu E \propto (1+z)$, where E is the energy of the neutrino beam. For relativistic background with energy E_b one gets $s \simeq 2E_b E \propto (1+z)^2$.

Let us consider neutrino propagation in a strongly CP-asymmetric background, $n_i \gg n_{\bar{i}}$, with $\eta_\nu \sim 1$. For simplicity we assume also flavour symmetry: $n_{\nu_e} = n_{\nu_\mu} = n_{\nu_\tau}$.

For the $\nu_\alpha - \nu_s$ channel the refraction width (68) increases smoothly from its low energy value, $d_0(s_Z \ll 1) = 3d_0/4$, to the high energy one, $d_0(s_Z \gg 1) = d_0$. For neutrinos produced with energy $E \lesssim 10^{21}$ eV and mass of the background neutrino $m_{\nu_\alpha} \simeq 2$ eV we get $s_Z \lesssim 0.5$, which undergoes redshift during the neutrino propagation. Therefore we can use the low energy value of d_0 as the effective refraction width. From (96) we get the ratio $r \equiv d/d_0(s_Z)$ in terms of the redshift z and the asymmetry η_ν :

$$r(z) \equiv \frac{4d(z)}{3d_0} \simeq 2.2 \cdot 10^{-2} \eta_\nu (z+1)^{\frac{3}{2}}. \quad (97)$$

Taking the maximal allowed asymmetry, $\eta_\nu \sim 10$, we find that $r = 3$ is reached at $z \simeq 5$, which corresponds to rather recent epoch. Possible sources of high energy neutrinos, the quasars, have been observed at such values of the redshift. With smaller asymmetries, $\eta_\nu \lesssim 1$, the minimum width condition requires much earlier epochs of neutrino production, $z \gtrsim 27$.

Notice that in general the minimum width condition, $d \geq d_0/\sin 2\theta$, is not sufficient to ensure a significant transition effect: as discussed in section 2, the width $d_{1/2}$ required to have conversion probability $P \geq 1/2$ depends on the specific effect involved. For the adiabatic conversion in varying density (section 2.3) we have found the result $d_{1/2} \simeq 1.5d_{min}$ (see eq. (30)). Therefore, the condition $P \geq 1/2$ requires larger values of $r(z)$, $r \gtrsim 4.5$. From eq. (97), with $\eta_\nu \sim 10$, we get $r \sim 4.5$ for $z \simeq 7.5$.

For $\bar{\nu}_\alpha - \bar{\nu}_s$ channel the refraction width d_0 , eq. (69), has a resonance character with absolute minimum $d_0(s_Z^{min}) \simeq d_0/7$ at $s_Z^{min} \sim 1$. Outside the resonance it takes the values $d_0(s_Z \ll 1) = 3d_0/4$ and $d_0(s_Z \gg 1) = d_0$. For neutrino energy $E \lesssim 10^{21}$ eV at production and mass of the background neutrino $m_{\nu_\alpha} \lesssim 1$ eV we get $s_Z < 1$, so that we can use the low energy value of the refraction width and the result for the ratio r coincides with that in eq. (97). For neutrinos produced at time $t = t_i$ with extremely high energies, $E \sim 10^{21} \div 10^{22}$ eV and mass of the background neutrino $m_{\nu_\alpha} \simeq 1 \div 3$ eV we get $s_Z \geq 1$ at the production time, so that, due to redshift, $s_Z(z)$ will cross the resonance interval, in which d_0 has minimum. This, however, does not lead to larger values of the ratio $r(z)$, since the time interval Δt during which s_Z remains in the resonance range is short: $\Delta t/t_i \simeq 1.5\gamma_Z \simeq 0.04$.

The matter width collected in the interval Δt is $d_{res} \simeq d\Delta t/t_i \simeq 1.5\gamma_Z d$, and the ratio $r(z)$ for the resonance epoch equals $r(z) = d_{res}/d_0(s_Z^{min}) \simeq d/4d_0$. That is even smaller than r outside the resonance, eq. (97); therefore the result (97) holds also in this case.

Thus, in the extreme condition of very large $\nu - \bar{\nu}$ asymmetry and production epoch at $z \gtrsim 5$ the matter width crossed by neutrinos satisfies the minimum width condition. Let us comment on the character of the matter-induced neutrino conversion in this case. After its production, the neutrino beam experiences a monotonically decreasing density. Moreover, the energy of neutrinos decreases due to redshift, which also influences the mixing. Taking into account the decrease with time of both the neutrino energy and concentration, the adiabaticity condition (18)-(19) can be generalized as:

$$\gamma(t) = \gamma_0 \frac{t}{t_0} = \gamma_0 (z+1)^{-\frac{3}{2}} \ll 1 \quad (98)$$

$$\gamma_0 \simeq \frac{8}{3V_0 t_0 \tan^2 2\theta} \ , \quad (99)$$

where V_0 is the present value of the neutrino-medium potential. For the present epoch we get $\gamma_0 \simeq 10^2/\tan^2 2\theta$, so that the adiabaticity is strongly broken. For $\tan^2 2\theta \lesssim 0.1$ the adiabaticity can be realized at $t/t_0 < 10^{-3}$, or $z > 10^2$. Thus, for $z < 10^2$ the conversion takes a character of oscillations in the production epoch. A detailed study of the dynamics of neutrino conversion in the universe will be given elsewhere [?].

For the flavour channel $\nu_\alpha - \nu_\beta$ the refraction effect appears at high energies, $s_Z \sim 1$, as a consequence of different energies of the background neutrinos ν_α and ν_β . The absolute minimum of the refraction width (71), $d_0^{min}(s_Z^i) = 3d_0$, gives a value of $r(z)$ which is 4 times smaller than that in eq. (97). Correspondingly, even for the extreme conditions of $\eta_\nu \simeq 10$ and $z = 5$ we get $r \simeq 0.8$. That is, significant matter effect is excluded for neutrinos from the oldest observed sources. The value $r \gtrsim 3$ can be reached for $z \gtrsim 14$. Notice that, according to eq. (71), the refraction disappears for low energies, $s_Z^i \ll 1$, and the redshift spoils the conditions of absolute minimum of d_0 , even if it is realized in certain epoch. Therefore, larger z are required to have significant conversion effect.

Similar conclusions can be obtained for $\bar{\nu}_\alpha - \bar{\nu}_\beta$ channel, where the refraction width (74) has the local minimum $d_0(s_Z^i) \simeq 6\gamma_Z d_0$ due to Z -resonance. This minimum is realized however during the resonance epoch Δt . Taking the corresponding matter width $d_{res} \simeq 1.5\gamma_Z d$, we get $r(z) = d(z)/4d_0$, which is even smaller than in the $\nu_\alpha - \nu_\beta$ case.

Let us consider CP-symmetric neutrino background, $n_i = n_{\bar{i}}$, with the assumption of flavour symmetry: $n_{\nu_e} = n_{\nu_\mu} = n_{\nu_\tau}$. As discussed in section 4.4, unsuppressed refraction effect appears for active-sterile neutrino channels at high energies, $s_Z \sim 1$, where the propagator corrections become important.

For $\nu_\alpha - \nu_s$ (and similarly for $\bar{\nu}_\alpha - \bar{\nu}_s$, due to CP-symmetry) the refraction width (76) has the absolute minimum $d_0(s_Z^{min}) \simeq 2\gamma_Z d_0$, realized at $s_Z^{min} = 1 \pm \gamma_Z$, eq. (77). Assuming that the neutrinos are produced just before the resonance epoch, we find that the width collected during the interval Δt equals $d_{res} \simeq d\Delta t/t_i \simeq 1.5\gamma_Z d$, where d is given

by eq. (96), with $d_U = 2n_\gamma^0 t_0 / 11 \simeq 7 \cdot 10^{29} \text{ cm}^{-2}$. For the ratio $r(z)$ we find:

$$r(z) = \frac{3d(z)}{4d_0} = 2 \cdot 10^{-3} (z+1)^{\frac{3}{2}}, \quad (100)$$

which is significantly smaller than $r(z)$ for $\nu_\alpha - \nu_s$ channel in CP-asymmetric background, eq. (97). With the maximal redshift $z \simeq 5$ eq. (100) gives $r \simeq 0.03$, which excludes matter effect for neutrinos from the most distant observable sources. The result (100) holds also for the $\nu_\alpha - \nu_\beta$ ($\bar{\nu}_\alpha - \bar{\nu}_\beta$) channel, since the refraction width (79) has analogous behaviour to the one for the $\nu_\alpha - \nu_s$ case, eq. (76), with the same local minimum $d_0(s_Z^i) \simeq 2\gamma_Z d_0$ at resonance.

In conclusion, we have found that the matter effect for neutrinos crossing the universe is mainly due to the neutrino background. For neutrinos from observable sources ($z \simeq 5$), significant conversion effect can be achieved in the $\nu_\alpha - \nu_s$ and $\bar{\nu}_\alpha - \bar{\nu}_s$ channels, if the background has strong CP-asymmetry, close to the maximum value, $\eta_\nu \simeq 10$. The matter effect for the other conversion channels and for the CP-symmetric case is suppressed as a consequence the redshift.

6 Conclusions

1). Matter effects can lead to strong flavour transition even for small vacuum mixing angle: $\theta \ll 1$. This however requires a sufficiently large amount (width) of matter crossed by neutrinos: the minimum width condition, $d \geq d_{min}$, should be satisfied, where $d_{min} = \pi / (2\sqrt{2}G_F \tan 2\theta) = d_0 / \tan 2\theta$, for low neutrino energies, $s \ll M_Z^2$, and conversion probability $P \geq 1/2$. The absolute minimum d_{min} is realized for uniform medium with resonance density n_e^{res} .

2). We have shown that for all the other realistic situations the required width, $d_{1/2}$, is larger than d_{min} . In particular, we have found that $d_{1/2}/d_{min} = 1 + (1 - \pi/8)\gamma^2$ for oscillations in medium with slowly varying density ($\gamma \ll 1$); $d_{1/2}/d_{min} \geq 1.5$ for conversion in medium with varying density; $d_{1/2}/d_{min} = \pi$ for castle wall profile.

3). We discussed the minimum width condition for high energy neutrinos. For $s \gtrsim M_W^2$ the minimum width d_{min} becomes function of s , due to the propagator effect: $d_{min} = d_{min}(s)$. The function $d_{min}(s)$ depends on the channel of interactions: in the case of W (or Z) exchange in the s -channel $d_{min}(s)$ decreases in the resonance region by a factor ~ 20 with respect to the low energy value: $d_{min}(0)/d_{min}(M_W^2) \sim 20$. In this region, however, the inelastic interactions become important, damping the flavour conversion.

4). As a case of special interest we have studied the refraction of high energy neutrinos in neutrino background, which can be important for propagation of cosmic neutrinos in galaxies and intergalactic space. Again we find that the $\nu_\alpha - \bar{\nu}_\alpha$ annihilation channel gives

enhancement of refraction at $s \simeq M_Z^2$, so that d_{min} can be $\sim 1/2\gamma_Z \sim 20$ times smaller than that at low energies. In the case of flavour channels the refraction can appear as the result of the difference of masses of the background neutrinos, even if the concentrations of the various flavours are equal.

5). The minimum width condition allows one to conclude on the relevance of the matter effect without knowledge of the density profile, once the width d is known. In some astrophysical situations the total width on the way of neutrinos can be estimated rather precisely (e.g. by spectroscopical methods) although the density distribution is unknown. Significant matter effect is excluded if $d < d_0$, or $d < d_{min}$ if the mixing angle is known.

6). From practical point of view, a study of the matter effects should start with the check of the minimum width condition, $d \geq d_{min}$. This condition is necessary but not sufficient for strong conversion effect. If it is fulfilled the ratio d/d_0 allows one to estimate the minimal mixing angle for which significant transition is possible: $\sin 2\theta > d_0/d$. This condition gives an absolute lower bound on θ , which can be achieved for the case of uniform medium with resonance density. In other words, given the width d of the medium, the highest sensitivity to the mixing angle is achieved if the matter is distributed uniformly and the density coincides with the resonance value for a given neutrino energy. For media with non-uniform matter distribution the sensitivity to θ is lower. The stronger the deviation from the constant density, the lower the sensitivity.

7). We applied the minimum width condition to neutrinos in AGN and GRBs environment. For AGN the width d can be estimated by the experimental data on the X-ray spectrum, without assumptions on the density profile. We got $d/d_0 \lesssim 10^{-10}$ for radially moving neutrinos, strongly excluding matter effects. In the case of GRBs the width d can be evaluated under the assumption that the object is transparent to protons. We found $d/d_0 \lesssim 10^{-5}$. Therefore, no significant conversion is expected either.

8). For neutrinos crossing the halos of galaxies and clusters of galaxies the matter effect is given by the interaction of neutrinos with the neutrino component of the halo. We have found that for galactic halos the minimum width condition is not satisfied: the result $d(halo)/d_0 \leq 0.1$ excludes any significant conversion effect. For halos of clusters of galaxies we got $d(halo)/d_0 \gtrsim 10$, and the minimum width condition can be satisfied for large enough mixing: $\sin 2\theta \gtrsim 0.1$.

9). We have considered the refraction of neutrinos from cosmologically distant sources, interacting with the neutrino background of the universe. Significant active-sterile conversion can be expected in case of large $\nu - \bar{\nu}$ asymmetry. We have found that for $\eta_\nu = O(1)$ the condition $d(universe)/d_0 \gtrsim 1$ can be achieved for neutrinos from sources, galaxies or quasars, with redshift $z \gtrsim 5$. The effect on detected neutrino fluxes from these sources could be a distortion of the energy spectrum.

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